

# Relative information encoded in the degree of entanglement to discriminate bipartite states

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## Abstract

It has been recently shown (Bartlett et al. 2003) that information encoded into relative degrees of freedom enables communication without a common reference frame using entangled bipartite states. In this case the relative information stored in the two-qubit system is shared between the polarization degrees of freedom and the degree of entanglement. In the present article a specific state discrimination problem is envisioned where the degree of entanglement carries the only relative parameter, so that certain maximally entangled states are perfectly distinguishable, while discrimination of product states is impossible.

## 1 Introduction

In a composite system, be it classical or quantal, made up of two or more distinct parts, one can distinguish between global and relative parameters. Global parameters belong to some properties of the whole system with respect to an external reference system, while relative parameters describe relations between the individual parts of the system. On the other, measuring the relative degrees of freedom of a system exhibits large differences between the classical and the quantal settings [1, 3]: in the classical world one cannot measure relative parameters without actually measuring the global

parameters as well, while in the quantum world it is possible to perform measurements revealing merely relative properties of the system. (Moreover, it has been found [2, 3, 4] that an optimal measurement of the relative angle between two 'quantum axes' made out of  $N_1$  and  $N_2$  particles can be performed without the need of an external reference frame.) This also implies that if one try to encode information in the quantal case in the relational degrees of freedom, this information will not depend on global effects acting on the system as a whole. Indeed, this kind of relative information is useful in quantum computing allowing to design noiseless quantum codes [5, 6, 7], in quantum cryptography permitting to distribute quantum key in the presence of arbitrary degree of collective noise [8, 9], or in quantum communication if no shared reference frame (Cartesian or Lorentz) exists between the communicating parties [10, 11, 12, 13].

On the other hand, relative information is also a useful resource in classical communication over a quantum channel if either the noise on the quantum channel has some partial correlations [14, 15, 16] or when the communicating parties do not share a common reference frame [10]. In this latter setting two qubits are employed to send a single classical bit of information: The sender transmits either the maximally entangled, antisymmetric singlet state  $|\psi^-\rangle$ , or any state from the symmetric subspace  $I - |\psi^-\rangle\langle\psi^-|$ . Then the receiver can distinguish the above two states with certainty by performing a projective measurement on the respective antisymmetric and symmetric subspaces. Since the state from the symmetric subspace can be a product state as well, it is suspected that entanglement does not only play the role of carrying data encoded into relative degrees of freedom. In fact in a general two-qubit system, according to Schmidt's theorem [17, 18], there are three relative parameters [2]: The polarization angle between the two spins in the Schmidt decomposition, the degree of entanglement and the phase between the two terms in this decomposition.

Thus, it seems an interesting question to raise whether it is possible to imagine a physical problem where out of the above three relative parameters only the degree of entanglement carries the whole information content in the bipartite system. The main purpose of the present article is to find an example of a state discrimination problem which fulfills this requirement, so that maximally entangled input states enable perfect state discrimination while product input states are completely indistinguishable (section 2). We also give a proof (in section 3) that in the given example the measured two-qubit states carry relative information encoded indeed only in the degree of entanglement, and that they can be distinguished with certainty. Then (in

section 4) this problem is examined in its most generality by calculating the performance of the measurement for non-maximally entangled and product states as well.

## 2 Searching for distinguishable states

Our problem is related to the discrimination of separated, unentangled two-qubit correlated states using non-local measurements [19]. In that case one party prepares two kind of correlated product states of spin-1/2 particles,  $|\psi_1^{sep}\rangle = |\mathbf{n}\rangle|-\mathbf{n}\rangle$  and  $|\psi_2^{sep}\rangle = |\mathbf{n}\rangle|\mathbf{n}\rangle$ , where  $|\mathbf{n}\rangle$  is chosen randomly and uniformly from the whole Bloch sphere. The task of the other party (i.e the observer) is to identify the above states with the best efficiency. Since the single qubit states  $|\pm\mathbf{n}\rangle$  are completely unknown for the observer the identification cannot be accomplished unambiguously. Actually, the information contained in these correlated product states is supported only by the relative angles (zero or  $\pi$ ) between the pair of spins. It has been shown [19], that in the optimal joint measurement the average probability  $\bar{P}$  to discriminate correctly  $|\psi_1^{sep}\rangle$  and  $|\psi_2^{sep}\rangle$  is  $\bar{P} = 3/4$ . On the other hand, if the input states were not restricted to product states and  $|\psi_1\rangle$  would be the antisymmetric state  $|\psi^-\rangle$ , while state  $|\psi_2\rangle$  would lie in the symmetric subspace  $I - |\psi^-\rangle\langle\psi^-|$ , one could perfectly distinguish between them by performing a projective measurement onto the symmetric and antisymmetric subspaces [10]. Thus the choice of the following states

$$|\psi_j\rangle = \frac{1}{\sqrt{2}}(|\mathbf{n}\rangle|-\mathbf{n}\rangle \pm |-\mathbf{n}\rangle|\mathbf{n}\rangle) \quad (1)$$

satisfies the criterion of perfect discrimination, where the plus sign stands for  $j = 1$  (the antisymmetric singlet state), while the minus sign represents  $j = 2$  (the symmetric states). It is noted, that the product states  $|\mathbf{n}\rangle|\mathbf{n}\rangle$  are also elements of the symmetric subspace. However, it is completely indifferent to the outcome of the measurement if we choose from the symmetric subspace either the product state  $|\mathbf{n}\rangle|\mathbf{n}\rangle$  or the maximally entangled state  $|\psi_2\rangle$  in Eq. (1). Hence there is no need that both distinguishable states be maximally entangled, indicating that in this case entanglement should be only partially responsible for the discrimination of the states  $|\psi_1\rangle$  and  $|\psi_2\rangle$  in Eq. (1) above (although these states are both maximally entangled). This means that relative information must be stored here partly in the degree of entanglement and partly in the relative angles between the spin directions of the pair of qubits.

Our main goal is to transform away the relative polarization degrees of freedom thereby leaving us merely with the degree of entanglement as the single relative parameter. In order to accomplish it, we propose the following new input states

$$|\psi_j^{re}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{n}^{re}\rangle|-\mathbf{m}^{re}\rangle \pm |-\mathbf{n}^{re}\rangle|\mathbf{m}^{re}\rangle), \quad (2)$$

where the plus (minus) sign represents  $j = 1$  ( $j = 2$ ), as before in Eq. (1), but now  $|\mathbf{n}^{re}\rangle$  and  $|\mathbf{m}^{re}\rangle$  are completely uncorrelated and are both chosen from a uniform distribution over the polar great circle, which is a circle lying in the  $x - z$  plane of the Bloch sphere. These states formally originate from the states defined in Eq. (1), by switching in the second qubit the polarization state  $|\mathbf{n}\rangle$  to  $|\mathbf{m}\rangle$  and picking  $|\mathbf{n}\rangle$  and  $|\mathbf{m}\rangle$  only from the polar line of the Bloch sphere (superscript *re* refers to the latter restriction that the corresponding states in the  $x - z$  plane are confined to the real vector space). Notice, that taking account of the complete randomness and lack of correlation between  $|\mathbf{n}^{re}\rangle$  and  $|\mathbf{m}^{re}\rangle$ , the states in Eq. (2) no longer support the relative parameters encoded in the polarization degrees of freedom. Therefore, states in Eq. (2) due to the lack of knowledge of the observer about the points  $|\mathbf{n}^{re}\rangle$  and  $|\mathbf{m}^{re}\rangle$  are represented by the mixed density operator

$$\rho_j = \int d\mathbf{n} d\mathbf{m} |\psi_j^{re}\rangle \langle \psi_j^{re}| \quad (3)$$

for  $j = 1, 2$ , where  $d\mathbf{n}$  and  $d\mathbf{m}$  are uniform measures on the polar line of the Bloch sphere.

The reason for the restriction to a smaller subset of  $|\mathbf{n}\rangle$  and  $|\mathbf{m}\rangle$  comes from the need to distinguish between  $\rho_1$  and  $\rho_2$  with perfect fidelity. Namely, taking the pair of input states in Eq. (2) with  $j = 1$  and  $j = 2$ , they are related by the unitary transformation  $\sigma_z \otimes I$ , where  $\sigma_z$  is written in the basis  $\{|\mathbf{n}^{re}\rangle, |-\mathbf{n}^{re}\rangle\}$ . However,  $\rho_j$  in Eq. (3) is invariant under the orthogonal transformation  $|\psi_j^{re}\rangle \rightarrow U_1 \otimes U_2 |\psi_j^{re}\rangle$ , where  $U_1, U_2 \in \text{SO}(2)$ . Since  $\sigma_z \notin \text{SO}(2)$ , this implies that  $\rho_1$  is not guaranteed to be equal to  $\rho_2$ . On the other hand, if  $|\mathbf{n}^{re}\rangle$  and  $|\mathbf{m}^{re}\rangle$  in Eq. (2) were allowed to sample the whole Bloch sphere (by having them taken to be  $|\mathbf{n}\rangle$  and  $|\mathbf{m}\rangle$ ) the invariance of  $\rho_j$  would extend to any unitary operator of the form  $U_1 \otimes U_2$ , eventually resulting in  $\rho_1 = \rho_2$ . We hope that, on the contrary, the states defined by Eq. (2) will be perfectly distinguishable. This conjecture will be proved and supported with further calculations in the next sections.

### 3 The case of subset of maximally entangled states

Using Schmidt's theorem the state of any pure two-qubit system, up to an irrelevant overall phase factor, may be written as

$$|\psi_j\rangle = e^{-i\beta_j/2} \cos \frac{\alpha_j}{2} |\mathbf{n}_j\rangle |\mathbf{m}_j\rangle + e^{i\beta_j/2} \sin \frac{\alpha_j}{2} |-\mathbf{n}_j\rangle |-\mathbf{m}_j\rangle, \quad (4)$$

where  $j = 1, 2$  denote the two distinct states to be discriminated from each other, and  $(\alpha_j, \beta_j)$  are the Schmidt parameters, left unchanged under local  $SU(2)$  transformations. Therefore, beside the spherical angle between  $|\mathbf{n}_j\rangle$  and  $|\mathbf{m}_j\rangle$ ,  $\alpha_j$  and  $\beta_j$  are relative parameters of the system too [2].

Note that due to the general treatment, from now on  $|\psi_j^{re}\rangle$  denotes a general two-qubit state defined by Eq. (4) with the restriction that  $|\mathbf{n}_j\rangle$  and  $|\mathbf{m}_j\rangle$  are taken from the polar great circle, in contrast to the definition by Eq. (2).

If we quantify the pure state entanglement by the concurrence  $C$ , which is an operationally well defined measure of entanglement on its own right [20], we obtain

$$C_j = 4\det(\text{Tr}_2|\psi_j\rangle\langle\psi_j|) = \sin^2 \alpha_j, \quad (5)$$

where  $\text{Tr}_2$  means tracing over the second qubit from the two-qubit state, and  $|\psi_j\rangle$  is an arbitrary state from Eq. (4). Thus the angle  $\alpha_j$  solely determines the degree of entanglement in an arbitrary pure two-qubit state, and it can be immediately read off from Eq. (5) that  $\alpha_j = \{0, \pi\}$  correspond to factorizable states, while maximally entangled states are obtained for  $\alpha_j = \{\pi/2, 3\pi/2\}$  (the latter  $\alpha_j$ 's also correspond to the entangled states defined by Eq. (2) in the previous section).

Let us calculate now the density operator  $\rho_j$  in Eq. (3) for  $j = 1, 2$  explicitly by taking the quantum states  $|\psi_j^{re}\rangle$  from Eq. (4) with the restriction that  $|\mathbf{n}_j\rangle, |\mathbf{m}_j\rangle$  represent random points on the polar great circle. Thus,  $\rho_j(\alpha_j, \beta_j) = \int d\mathbf{n} d\mathbf{m} |\psi_j^{re}\rangle\langle\psi_j^{re}|$  in this general case also does not depend on the relative parameter carried by  $|\mathbf{n}^{re}\rangle$  and  $|\mathbf{m}^{re}\rangle$ , and  $\rho_j$  becomes a function of only the pair of parameters  $(\alpha_j, \beta_j)$ .

By performing the integration in Eq. (3) for the input state  $|\psi_j^{re}\rangle$ , we obtain in the standard computational basis (i.e. in the basis of eigenstates of  $\sigma_z \otimes \sigma_z$ ) the following density matrix

$$\rho_j = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & -q_j \\ 0 & 1 & q_j & 0 \\ 0 & q_j & 1 & 0 \\ -q_j & 0 & 0 & 1 \end{pmatrix}, \quad (6)$$

where  $q_j = \sin \alpha_j \cos \beta_j$  for  $j = 1, 2$  and thus  $q_j \in [-1, 1]$ . After diagonalization one arrives at

$$\rho_j = \frac{1}{4}(1 - q_j) [|\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-|] + \frac{1}{4}(1 + q_j) [|\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+|] , \quad (7)$$

where the four Bell states  $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|+z\rangle|-z\rangle \pm |-z\rangle|+z\rangle)$ ,  $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle \pm |-z\rangle|-z\rangle)$  appear in the density matrix, and  $|+z\rangle, |-z\rangle$  represent points on the north and south poles, respectively. It is noted that  $\rho_j$  in Eq. (7) is completely determined by the value of  $q_j$ . Since  $\rho_j$  commutes with any operator of the form  $U_1 \otimes U_2$ , where  $U_1, U_2$  are  $\text{SO}(2)$  orthogonals (see Eq. (3)), this also applies to the particular case when  $q_j = -1$  or  $q_j = +1$ . Let us denote the two-dimensional subspaces corresponding to  $q_j = -1$  and  $q_j = +1$  in Eq. (7) by

$$\begin{aligned} E_1 &= |\phi^+\rangle\langle\phi^+| + |\psi^-\rangle\langle\psi^-| \\ E_2 &= |\phi^-\rangle\langle\phi^-| + |\psi^+\rangle\langle\psi^+| , \end{aligned} \quad (8)$$

respectively. Then  $\rho_j$  in Eq. (7) looks as

$$\rho_j = \frac{1}{4}(1 - q_j)E_1 + \frac{1}{4}(1 + q_j)E_2 , \quad (9)$$

and any state  $|\psi_j^{re}\rangle$  in the support of subspaces  $\{E_1, E_2\}$  is mapped by Eq. (3) to the completely mixed state  $\frac{1}{2}I_2$  over the respective two dimensional subspaces. This implies, following the argumentation of Bartlett et al. [2], that in this case the most informative measurement to estimate  $|\psi_j^{re}\rangle$  is simply the projection of these states onto the subspaces  $\{E_1, E_2\}$ . Hence, in our problem the POVM elements [18] of the optimal measurement are the projectors  $\{E_1, E_2\}$ .

Thus, provided that  $|\psi_1^{re}\rangle$  is chosen from the  $E_1$  subspace (i.e.  $q_1 = -1$ ), while  $|\psi_2^{re}\rangle$  is chosen from the  $E_2$  subspace ( $q_2 = +1$ ), performing a projective measurement on the subspaces  $E_1$  and  $E_2$  will exactly discriminate  $|\psi_1^{re}\rangle$  from  $|\psi_2^{re}\rangle$ . Note, that  $q_j = \pm 1$  implies  $\sin \alpha_j = \pm 1$  (i.e. the concurrence  $C_j$  in Eq. (5) is equal to 1), hence the set of the perfectly distinguishable states is maximally entangled. This result can also be obtained by observing that all the states which are in the support of  $\{E_j\}$  can be produced from one of the Bell states by performing local  $\text{SO}(2)$  rotations, however, rotations acting on the local Hilbert spaces of each qubit cannot change the degree of entanglement (which is maximal for the Bell states).

## 4 The general situation

In this section we discuss the case when the two-qubit states  $|\psi_j^{re}\rangle$  for  $j = 1, 2$  (taken from a restricted state space of the Schmidt decomposition in Eq. (4)) describe non-maximally entangled states. Let us introduce the following payoff function

$$\bar{P} = p_1 \text{Tr}[E_1 \rho_1] + p_2 \text{Tr}[E_2 \rho_2] , \quad (10)$$

which gives the measure of success of our measurement. Here  $p_j$  is the prior probability that  $|\psi_j^{re}\rangle$   $j = 1, 2$  is prepared, which can be taken  $p_1 = p_2 = 1/2$ , assuming that the observer has no prior knowledge of  $p_j$ . Substituting the explicit values of  $\rho_j$ ,  $E_j$  from Eq. (9) and Eq. (8) and  $p_j$  from above into the payoff function  $\bar{P}$ , one obtains

$$\bar{P} = \frac{1}{2} + \frac{\cos \beta_1 \sin \alpha_1 - \cos \beta_2 \sin \alpha_2}{4} . \quad (11)$$

Starting from this compact formula a brief elementary calculation shows, that for a given pair of  $(\alpha_1, \alpha_2)$   $\bar{P}$  is maximized for the states

$$|\psi_j^{re}\rangle = \cos \frac{\alpha_j}{2} |\mathbf{n}\rangle |\mathbf{m}\rangle \pm \sin \frac{\alpha_j}{2} |-\mathbf{n}\rangle |-\mathbf{m}\rangle , \quad (12)$$

where the signs plus/minus stand for  $j = 1, 2$ , meaning that the optimal states  $|\psi_j^{re}\rangle$  are confined to the real state space. In this case the average payoff  $\bar{P}$  is the single function of the degree of entanglement for the two distinct states and looks as follows

$$\bar{P} = \frac{1}{2} + \left| \frac{\sin \alpha_1 + \sin \alpha_2}{4} \right| . \quad (13)$$

We arrived at the main result of our specific parameter estimation problem: For spin-1/2 pairs restricted to the real vector space the average payoff  $\bar{P}$  of the measurement depends only on the degree of entanglement (characterized by  $\sin \alpha_j$ ,  $j = 1, 2$ ). Especially, for product states  $\bar{P} = 1/2$ , meaning that no information can be gained from the measurement to discriminate the two given product states from each other. Therefore, we are allowed to say, that in our state estimation problem, wherein the prepared states are real valued spin-1/2 pairs, only the degree of entanglement as relative parameter is responsible for acquiring information about the prepared states.

It has been shown in Ref. [21] that for states with the same symmetry properties as for  $\rho_j$  in Eq. (6) the separability criterion takes the form

$$\max\{|2f_j - 1| - 1, |2g_j - 1| - 1\} \leq 0 , \quad (14)$$

with  $f_j = \frac{1}{2}(1 + q_j)$  and  $g_j = \frac{1}{2}(1 - q_j)$ , where the above explicit values have been obtained by establishing a complete correspondence between the system considered in section V.B. of Ref. [21] and our state  $\rho_j$  in Eq. (6). Since the above inequality is satisfied for any parameters of  $(\alpha_j, \beta_j)$  the states  $\rho_1$  and  $\rho_2$  are both separable. However, it is interesting to mention that in the joint measurement  $\{E_1, E_2\}$  one could achieve perfect discrimination of maximally entangled pure states, despite the fact that the measured states  $\rho_j$  for  $j = 1, 2$  are always unentangled. This perfect discrimination would not have been possible if the measurement were only a local one, performed on each qubit separately even with the aid of classical communication.

## 5 Conclusion

In section 2 of this work we found a special parameter estimation problem for two-qubit systems where the degree of entanglement supports the single parameter of the system. In section 3 it has been proven that choosing quantum states from a special subset of the Bell states, perfect state discrimination is possible. Then, in section 4 a payoff function has been defined and with the aid of it we characterized the efficiency of the measurement in estimating spin-1/2 pairs restricted to the real vector space.

Let us mention a duality between the problem of the present work and the problem of Ref. [19]. In both cases only one single parameter carries information about the identity of the states to be estimated. In our problem the parameter is the degree of entanglement, while in Ref. [19] this parameter pertains to the angle between the spin directions in the product state of the pair of particles. In both cases the parameters are independent of global properties of the systems, thus they are called relative parameters. However, one difference may appear in the 'performance' of these relative parameters: while in our task an optimal joint measurement can perfectly discriminate maximally entangled states, on the other hand product states in the problem of Ref. [19] are cannot be distinguished with perfect fidelity.

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## References

- [1] N. Gisin, S. Iblisdir, arXiv: quant-ph/0507118
- [2] S.D. Bartlett, T. Rudolph, and R.W. Spekkens, Phys. Rev. A **70** 032321 (2004)
- [3] E. Bagan, S. Iblisdir, R. Munoz-Tapia, arXiv: quant-ph/0508187
- [4] N.H. Lindner, P.F. Scudo, and D. Bruss, arXiv: quant-ph/0506223
- [5] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79** 3306 (1997)
- [6] M.S. Byrd, D.A. Lidar, L.-A. Wu, P. Zanardi, Phys. Rev. A **71** 052301 (2005)
- [7] T. Rudolph and S.S. Virmani, arXiv: quant-ph/0503151
- [8] J.-C. Boileau, D. Gottesman, R. Laflamme, D. Poulin, and R.W. Spekkens, Phys. Rev. Lett. **92** 017901 (2004)
- [9] Z.D. Walton, A.F. Abouraddy, A.V. Sergienko, B.E.A. Saleh, and M.C. Teich, Phys. Rev. Lett. **91** 087901 (2003)
- [10] S.D. Bartlett, T. Rudolph, and R.W. Spekkens, Phys. Rev. Lett. **91** 027901 (2003)
- [11] M. Bourennane, M. Eibl, S. Gaertner, C. Kurtsiefer, A. Cabello, and H. Weinfurter, Phys. Rev. Lett. **92** 107901 (2004)
- [12] S.J. van Enk, arXiv: quant-ph/0602079
- [13] S.D. Bartlett, D. R. Terno, Phys. Rev. A **71** 012302 (2005)
- [14] C. Macchiavello and G.M. Palma, Phys. Rev. A **65** 050301 (2002)
- [15] J. Ball, A. Dragan, and K. Banaszek, Phys. Rev. A **69** 042324 (2004)
- [16] K. Banaszek, A. Dragan, W. Wasilewski, and C. Radzewicz, Phys. Rev. Lett. **92** 257901 (2004)
- [17] E. Schmidt, Ann. Math. **63** 433 (1907)
- [18] A. Peres, *Quantum Theory: Methods and Concepts* (Kluwer Academic, Dordrecht, 1995), pp. 123-126

- [19] G.J. Pryde, J.L. O'Brien, A.G. White, and S.D. Bartlett, Phys. Rev. Lett. **94** 220406 (2005)
- [20] W.K. Wootters, Phys. Rev. Lett. **80** 2245 (1998)
- [21] G. Vidal, R.F. Werner, Phys. Rev. A **65** 032314 (2002)